Problem Set 3
Physics498:Theoretical Biophysics /Spring 2001
Professor Klaus Schulten

Note on Term Papers:

In order to schedule appointments with Prof. Schulten to discuss the term projects chosen by you please send email to *markus@ks.uiuc.edu* with your preferred time and the project you have chosen. The available days are Thursday (03/29), Friday(03/30) or next week from 8pm-10pm in Prof. Schulten's office (Beckman Rm 3147).

Problem 1: Verhulst Equation

Consider the evolution equation

$$\dot{x} = x - x^2 \tag{1}$$

that describes the potential $V(t) = x$ of a neuron.

(a) Determine the stationary points x_{1s}, x_{2s} and the behaviour of the solution in linear approximation x_{1s}, $x_{2s} = a$, $\delta x = x - a$, $f(x) = x - x^2$

$$\delta \dot{x} = f(a) + \partial_x f|_a \delta x \tag{2}$$

(b) Derive the exact solution of eq.(1) for arbitrary initial condition $x(0) = x_0$. Compare the result with the approximation considered in (a).
(c) For a numerial solution of eq. (1) one discretizes the time-derivative in eq.(1) through $(x_{n+1} - x_n)/\Delta t$ for $t \to t_n = n\Delta t$ and $x_n = x(t_n)$. Assume that the differential equation (1) is replaced by

$$x_{n+1} - x_n = h(x_{n+1} - x_n x_{n+1}) \tag{3}$$

Show that the solution of this equation is

$$x_n^{-1} = 1 - (1 - x_0^{-1})(1 - h)^n \tag{4}$$

Discuss the behaviour of this solution for $h > 0$, $h \to 0$ and for $h < 1$ and $h \geq 1$.

Problem 2: Limit Cycle

Discuss the solution of the equation

$$\begin{aligned} \dot{x} &= y + xf(r) \\ \dot{y} &= -x + yf(r) \end{aligned} \tag{5}$$

$(r = \sqrt{x^2 + y^2})$ for the functions $f(r) = (1 - r^2)$, $(r^2 - 1)$ and $(r^2 - 1)^2$.
(Hint: Use polar coordinates).

Problem 3: Bonhoefer-van der Pol Equation

Consider the solution of the evaluation equation $(c = 3, \; a = 0.7, \; b = 0.8)$

$$\begin{aligned} \dot{x}_1 &= c\left(x_2 + x_1 - \frac{x_1^3}{3} + z\right) = f_1(x_1, x_2) \\ \dot{x}_2 &= -\frac{1}{c}(x_1 + bx_2 - a) = f_2(x_1, x_2) \end{aligned} \tag{6}$$

(a) Discuss the stationary point $\mathbf{x}_s$ of this system of equations in the range $-2 \leq z \leq 2$, i.e $\mathbf{f}(\mathbf{x}_s) = 0$.

(b) For $z = 0$ and $z = -0.4$ carry out a linear stability analysis, i.e. solve the equation $(\delta \mathbf{x} = \mathbf{x} - \mathbf{x}_s)$

$$\delta \dot{\mathbf{x}} = \mathbf{M} \delta \mathbf{x} \tag{7}$$

for $M_{jk} = \partial_k f_j(\mathbf{x}_s)$. For this purpose expand the solutions in terms of the eigenvectors of $\mathbf{M}$.
Note: The eigenvectors for different eigenvalues may not be orthogonal!

(c) Determine typical trajectories $\mathbf{x}(t)$ for $z = 0$ and $z = -0.4$ for a variety of initial points $\mathbf{x}(0)$. For this purpose discretize the system eqs. (6) in the form

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t f(\mathbf{x}_n) \tag{8}$$

for suitable Δt. Plot the result.

(d) Consider now eq. (8) in the form

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t f(\mathbf{x}_n) + \sigma \sqrt{\Delta t} \begin{pmatrix} \zeta_1(n) \\ \zeta_2(n) \end{pmatrix} \tag{9}$$

for $\sigma = 0.15$ and for random numbers $\zeta_1(n)$, $\zeta_2(n)$ that are Gaussian distributed, i.e. $p(\zeta_i) = 1/\sqrt{2\pi} \exp[-\zeta_i^2]$. Plot the results.

Problem 4: Cable Equation

Solve the cable equation

$$(\partial_t - \partial_x^2 + 1)v(x,t) = 0 \qquad (10)$$

for $x \in [0,a]$, the boundary conditions

$$\partial_x v \;=\; 0 \;, \quad x = 0 \qquad (11)$$
$$v(a) \;=\; 0 \qquad (12)$$

and the initial condition

$$v(x,0) = \delta(x) \;. \qquad (13)$$

For this purpose expand the solutions in terms of eigenfunctions of ∂_x^2 that obey the boundary conditions, i.e.

$$f_n(x) = \sqrt{\frac{2}{a}} \cos \frac{\pi n x}{2a} \qquad (14)$$

$n = 1, 3, 5, ...$, i.e. set $v(x,t) = \sum_{n=1,3,5,...} \alpha_n(t) f_n(x)$. Determine the $\alpha_n(t)$ that obey the initial condition eq. (13). Plot the result.

Due Thursday April 5th in class